On Eigenvalue Decompactification in QCD_{1+1}

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The question whether it is necessary to decompactify the gauge eigenvalue degrees of freedom in QCD_{1+1} is addressed. A careful consideration of the dynamics governing these degrees of freedom leads to the conclusion that eigenvalue decompactification is not necessary due to the curvature on the space of eigenvalues.

Keywords: QCD in one space dimension, Quantization on curved manifolds.

Recently, concerns have been raised regarding a possible requirement of "eigenvalue decompactification" [1] [2] in QCD in one space and one time dimension (QCD_{1+1}). The emergence of such a requirement would e.g. in the limit of a large number of colors cast serious doubts upon the validity of the usual large N counting arguments [2]. The purpose of this note is to reexamine carefully the dynamics governing the abovementioned eigenvalues; this will lead to the conclusion that eigenvalue decompactification is not necessary.

In order to familiarize the reader with the issue, a short review of the methodology of [1], [2] is in order. [1], [2], along with many other modern treatments of 1+1-dimensional gauge theories, choose the spatial coordinate to be compactified to a ring with tunable circumference L in order to obtain good control of the infrared properties of the model. On such a manifold, the gauge degrees of freedom cannot be completely gauged away; one must keep e.g. the zero momentum mode of the spatial component of the gauge field A_1. One may further constrain this mode to be diagonal, which leaves one with $N-1$ quantum mechanical gauge degrees of freedom in an $SU(N)$ theory [3]; these are the eigenvalues alluded to above (one of the eigenvalues, say in the following the Nth, is constrained due to the tracelessness of the $SU(N)$ generators). In addition, one has the original fermion fields as physical variables. The Hamiltonian can be written as

$$H = H_{KIN} + H_F + H_{COUL} \tag{1}$$

where H_{KIN} is the kinetic energy of the eigenvalues, H_F is the usual kinetic piece for the fermions, minimally coupled to the eigenvalues, and H_{COUL} is the Coulomb interaction, which also depends on the eigenvalues [3].

When quantizing the theory, one must take care to properly account for the curvature on the space of the eigenvalues. The measure in the scalar product of the Hilbert space of eigenvalue wave functions is the eigenvalue part of the Haar measure of $SU(N)$,

$$J = \prod_{i<j} \sin^2\left(\frac{\lambda_i - \lambda_j}{2}\right) \tag{2}$$

where the λ_i are the N eigenvalues of A_1 [4] [5]. This is easy to understand: Since in the gauge sector, only the momentum zero mode of the vector field is relevant, QCD_{1+1} can inherently be thought of as a one-link Hamiltonian lattice gauge theory (with periodic boundary conditions) [6]. The gauge degree of freedom parametrizes the link variable U in the canonical fashion[1],

$$U = e^{iA_1} \tag{3}$$

i.e. it parametrizes the $SU(N)$ rotation of the color frame of reference if one moves once around the spatial circle. Note that the fermions, on the other hand, may be allowed to move continuously along the link since, in one dimension, there is no path ordering ambiguity; a Wilson line is completely specified by its endpoints (up to the number of times

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[1]The gauge degree of freedom A_1 used here is already rescaled by a factor gL with respect to the A_1 one usually writes in the QCD Lagrangian.

it winds around the spatial circle). Specifically, if one moves a distance x in space, the color frame of reference is rotated by[2]

$$U_x = e^{iA_1 x/L} \tag{4}$$

Thus it becomes clear that the appropriate measure for the gauge degree of freedom is the Haar measure corresponding to the gauge group. Choosing a gauge in which A_1 is diagonal quantum mechanically corresponds to restricting the Hilbert space to wave functions independent of the angular parts of A_1, leaving only the eigenvalue part (2) as the relevant measure on the space of physical states.

Corresponding to the measure (2), the Laplacian entering the kinetic energy H_{KIN} acquires a Jacobian [4],

$$\Delta = \sum_{i=1}^{N-1} \frac{1}{J} \frac{\partial}{\partial \lambda_i} J \frac{\partial}{\partial \lambda_i} \tag{5}$$

Having established the form of the operators, one can now focus on the range of values the λ_i may take. The pure gauge theory is completely periodic in the eigenvalues, with period 2π; thus, one may think of this theory as a system of $N-1$ particles living on a circle. On the other hand, when fermions are included, the theory is only symmetric if, simultaneously to shifting an eigenvalue by 2π, one gives the appropriate phase to the fermions. Thus, in general, one will have to allow for an infinite range of values for the λ_i. This is what happens e.g. in the Schwinger model [8], where one must superimpose vacua related by large gauge transformations to form the well-known θ-vacua.

Now, however, one must take into account the Jacobian on the Hilbert space of eigenvalue wave functions discussed above[3]. This Jacobian prevents quantum mechanical propagation past the points where it is zero. One can see this clearly e.g. by constructing the probability current for the λ_i in the usual way, by demanding conservation of probability [11]. The probability current is proportional to J, and thus vanishes whenever two of the N eigenvalues meet modulo 2π. Since furthermore, their center of mass is fixed, they cannot propagate out of a certain compact fundamental domain. Note that this argument remains valid when taking into account the fermions, as long as the Hamiltonian remains local in the space of eigenvalues and does not contain attractive singular potentials which could induce the eigenvalues to fall towards each other. Both of these conditions are fulfilled. It should be mentioned that the Coulomb interaction indeed contains quadratic singularities at the zeros of the Jacobian [1] [7]; they however always have a positive coefficient, essentially the square of the fermionic color charge. Thus, except for fermionic color charge zero states, the effect of the Jacobian is even reinforced.

The different fundamental domains are consequently dynamically decoupled, there is no possibility of quantum mechanical interference like in the Schwinger model, which contains no Jacobian[4]. One can give a complete basis of eigenfunctions of the full Hamiltonian in terms of wave functionals which vanish on all but a single fundamental domain in the eigenvalues. Furthermore, there is no observable in the theory which connects different fundamental domains[5]. This means that, though one may define the eigenvalues to have an infinite range to begin with, one is in fact adding up trivial copies of the theory defined on one fundamental domain. One may just as well restrict to one of these domains only, factoring out the (infinite) number of domains; invariance under the corresponding residual gauge transformations simply ceases to be an issue.

In concluding, three further comments are in order: First, the considerations above were for an $SU(N)$ theory, which fixes the center of mass of the eigenvalues. If one considers a $U(N)$ theory, the center of mass is a further degree of freedom. This degree of freedom, and it alone, is not constrained to a compact domain by the Jacobian. Thus, one obtains the same $U(1)$ anomaly as in the Schwinger model. However, for all the relative coordinates, i.e. the eigenvalues with the center of mass subtracted off, the same discussion as above applies. The subtleties of the

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[2]A nice example of the intricate interplay between the gauge phases picked up by the fermions and the eigenvalue degrees of freedom is given in [7].

[3]The importance of such Jacobians has also been emphasized e.g. in [9], [10].

[4]Correspondingly, whereas the Schwinger model contains a nontrivial continuous θ-angle, which can be interpreted as the Bloch momentum of propagation around the A_1 circle, in the $SU(N)$ theory the Bloch momentum disappears due to the Jacobian barriers. This was first pointed out in [12]. Note that for quarks in the adjoint representation, there are actually additional symmetries within the fundamental domain which allow the construction of N different θ-vacua. However, one can never obtain a continuous band of θ-vacua [12] as would be produced if the shifts of the eigenvalues by 2π were nontrivial.

[5]In particular, the generator of translations of the variable λ_i, namely $i\partial/\partial\lambda_i$, is not hermitian in the presence of the measure (2); therefore, the quantum mechanical generator of the corresponding residual gauge transformations, which in the case of the Schwinger model has the θ-angle as its eigenvalue, is not an observable in the present case.

$N \to \infty$ limit connected with the question whether one includes the $U(1)$ part in the gauge group or not have been discussed e.g. by D.Stoll [13].

Secondly, the Jacobian (2) is often treated in the following way: By going to "radial" wave functions

$$\psi(\vec{\lambda}) = \prod_{i<j} \sin\left(\frac{\lambda_i - \lambda_j}{2}\right) \phi(\vec{\lambda}) \tag{6}$$

one may eliminate the Jacobian from the scalar product and the Laplacian (up to an irrelevant overall constant in the energy). The only, however important, relic of the curvature on the space of eigenvalues lies in the fact that $\psi(\vec{\lambda})$ now must vanish at certain points. Then the N eigenvalues are interpreted as fermionic degrees of freedom[6] [5] (with fixed center of mass in the $SU(N)$ case), since interchange leads to a minus sign in (6)[7]. There is nothing wrong with such an interpretation; it is one particular way of superimposing wave functions in different fundamental domains. One should however not be misled into thinking that this is the only legitimate way. Equally possible is e.g.

$$\psi(\vec{\lambda}) = \sqrt{J}\ \phi(\vec{\lambda}) \tag{7}$$

Note that the additional δ-functions one picks up when acting with the kinetic energy on (7) as opposed to (6) always coincide with zeros of the wave function. Thus, whether one chooses (6), (7), or restricts to a fundamental domain, one will always recover the same unique answer for physical quantities.

Finally, it should be noted that the conclusions reached above can also be made plausible in the path integral formalism. Consider for simplicity $SU(2)$, where there is only one independent eigenvalue λ. The points $\lambda = n\pi$ with integer n where the Jacobian vanishes correspond to poles of the four-dimensional sphere to which $SU(2)$ is isomorphic. Thus, heuristically, a path which leads past a zero of the Jacobian would have to lead exactly over a pole of the sphere. The set of such paths has measure zero. Formally, the Jacobian enters as integration measure of the λ_i at each time slice. Therefore, any path which leads over a zero of the Jacobian gives a vanishing contribution to the propagator.

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[6]Note that this contains all the information about the nodes of the wave function only as long as one has the eigenvalues living on a circle, since the Jacobian vanishes always when two eigenvalues coincide modulo 2π. If one allows an infinite range for the eigenvalues, one must additionally remember to impose nodes when two eigenvalues match up to a multiple of 2π. This crucial point was neglected in [2].

[7]Note that $\phi(\vec{\lambda})$ can be argued to be symmetric in the λ_i, reflecting the fact that it should not matter in which order one puts the eigenvalues when diagonalizing A_1.

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